An Interpretation of the Dimensions of the Pyramids of Gizah

I. Precession of the Axis of the Earth

1. In 1925 T. H. Cole, the topographer who established the modern system of triangulation for the Egyptian state, published a report on the dimensions of the base of the Great Pyramid of Gizah. Cole undertook his survey under the impulsion of the Egyptologist Ludwing Borchardt, who thought that an accurate report would help in separating facts from fiction in the matter of the geometry of the Pyramid. But no effort has thus far been made to exploit the precision achieved by Cole.

Cole reports the following data for the length of the sides:

West	230.357
South	230.454
East	230.391
North	230.253

The earlier survey conducted by Sir Flinders Petrie had ended with uncertain results; Petrie had difficulty in delimiting the exact position of the corners, due to the disappearance of many of the blocks. By an extensive sounding of the foundations, Cole was able ascertain the position of the corners with reasonable assurance: he computes as follows the maximum possible error in his values for the length of the sides:

North side	6 mm.	at either end
East side	6 mm.	at either end
South side	10 mm.	at the West end
	30 mm.	at the East end
West side	30 mm.	at either end

It is generally agreed that the lineal standard used in the construction of the Great Pyramid is the Egyptian royal cubit of 525 mm. This cubit is based on a foot of 300 mm. (which is the basic standard of all ancient metric systems) and on the corresponding cubit of 450 mm. From the cubit of 450 mm. there was derived a special cubit of 525 mm. called royal cubit; the royal cubit is an instance of a septenary unit of length, because it is composed of 7 hands or 28 fingers, whereas the normal cubit is divided into 6 hands or 24 fingers.[1] The royal cubit was the standard most frequently employed in Pharaonic Egypt and its use is documented since predynastic times.

2. It is generally agreed among scholars that the correct value of the Egyptian royal cubit is 525 mm., but it has also been noted that often buildings and measuring rods indicate a value of about 524 and value of almost 527. These values do not result from an imprecision of the standard, but have a specific metrological explanation.

In 1888 the Egyptologist Heinrich Brugsch, following the ideas developed by Pietro Bortolotti, came to the realization that the Egyptian unit of weight called *qdt* (*kite* in Coptic) was the basic unit of all ancient systems of weight, a unit which I call basic sheqel and compute as 9 grams. This fundamental discovery was further

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1. I have determined that septenary units of length were commonly used in Mesopotamia, Egypt, Greece and medieval Europe, because they allow to solve in a simple way problems of practical measurement involving the irrational roots π, $\sqrt{2}$, and $\sqrt{3}$. By using septenary units the circumference can be easily computed as 22/7 of the diameter (π = 3 1/7 = 3.1428). The diagonal of a square was computed by assuming that the square with a side 7 has a diagonal 10 or that a square with side 10 has a semi-diagonal 7. In the first case $\sqrt{2}$, which is 1.41421, was computed as 1.42857 and in the second case as 1.40; when greater precision was desired the result of the two computations was averaged arriving at a value $\sqrt{2}$ = 1.41428, which is correct to the fourth decimal point. In the case of an equilateral triangle, it was assumed that if the side was 7, the height was 6 (6/7 =0.85714, whereas the correct ratio is $\frac{1}{2}\sqrt{3}$ = 0.86602).

developed by Friedrich Hultsch in a special monograph that represents the conclusion of his life-long research on ancient measures. Brugsch observed that in Egypt the basic *sheqel* was computed in three different ways:

1) as 1/3000 of the cube of the foot of 300 mm. filled with water;
2) as 1/10,000 of the cube of the cubit of 450 mm.; and
3) as 1/16,000 of the cube of the royal cubit of 525 mm.

He remarked that in the first case the *sheqel* would be 9 grams, whereas in the second case it would be 9.1125 grams and in the third case it would be 9.0439 grams. Neither Brugsch nor Hultsch succeeded in explaining away these small differences.[2]

The mathematical explanation of this discrepancy was provided by Angelo Segrè. Jean Adolphe Decourdemanche, in the concluding page of his treatise on ancient and Arab metrology, observed that ancient and Arab units of weight have set values that vary as 80:81. Segrè observed that this discrepancy is a regular occurrence in the metrics of Greek papyri and results from the fact that the cube of the cubit is equal to 3⅜ cubes of the foot, since $1^3:(1\frac{1}{2})^3=1:3\frac{3}{8}$, but the relation was frequently simplified to a relation 1:3⅓.[3] The cube of

2. Brugsch remarked that Richard Lepsius, by examining Egyptian sample weights, had come to the conclusion that the *qdt* had a value of about 9.0591, and observed that accordingly the foot should be 301.06 mm.; but he did not pursue the problem any further. Hultsch also avoided the problem by using in one work the value of 9.1125 grams, equal to 1/10,000 of the cube with an edge of 450 mm., and in another work the value of 9.060 grams, which is simply a rounding of the figure of Lepsius. Petrie, who studied ancient weights without considering their relation to length, concluded that Egyptian sample weights indicate a *qdt* varying between 138 and 141 English grains (8.942 to 9.136 grams), with a clear indication of the existence of a variety of 140 grains (9.072 grams).
3. Hence, the cube of the cubit may be conveniently computed as 10 Roman modii of 16 sextarii or basic pints.

the cubit contains 162 basic pints of 540 cc., but both metrological treatises and economic documents often round the figure to 160 pints. The discrepancy 80:81 is a general occurrence in ancient metrics; I shall have occasion to demonstrate that the interval *komma* of the musical scales originated from it, since musical scales derive from the units of volume.

Ancient units of weight are at times computed by a basic sheqel of 9 grams and at times by a basic *sheqel* of 9.1125 grams.[4] In Egypt the intermediary value of 9.0439 grams was arrived at by computing from the royal cubit of 525 mm. But since the values of 9 and 9.1125 grams were also in use, there were concurrently used three forms of the royal cubit:

- royal cubit of 524.1483 mm. corresponding to a *qdt* of 9 grams
- royal cubit of 525 mm. corresponding to a *qdt* of 9.0439 grams.
- royal cubit of 526.564 mm. corresponding to a *qdt* of 9.1125 grams.

The cubit used in the construction of the Great Pyramid is of the first type. The best datum about the standard used is provided by the dimensions of so-called King's Chamber which, as Newton properly concluded, was calculated as 20 x 10 cubits. From its dimensions Newton inferred that the Egyptian royal cubit is equal to 1719/1000 of English foot, or 523.9507 mm. But Newton's computation should have given a figure of 1718.5 thousandths, a figure which he rounded to 1719. Newton based himself on the survey of John Greaves who

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4. For instance, the Roman *libra* is 324 grams, equal to 12 ounces of 27 grams or 3 basic *sheqels*, but in the age of the Emperor Vespasian there appears also one of 328.05 grams. This unit is the standard Roman *libra* of the Middle Ages; correspondingly in the Middle Ages the most correct Roman foot is considered that of 297.761 mm., usually called geometric foot.

employed a ruler copied on the standard of Guild Hall in London (*pes Curiae Londinensis*); I have computed the English foot of this standard as equal to the present American foot 304.8 mm., but it may have been some hundredths of millimeter more. By most accurate tests Petrie concluded that the cubit of the King's chamber was 524.052 + 0.10 mm. and that this was the standard of the Great Pyramid. In may opinion the theoretical value of the standard was 524.148 mm.

3. Since it is agreed among responsible scholars that the side of the Pyramid was computed as 440 royal cubits, its theoretical length was 230.625 mm. Hence, according to Cole's figures, the West side is shorter by 268 mm., the South side by 171, and the East side by 234. In my opinion the North side was made deliberately shorter by about $2\frac{2}{3}$ hands or 200 mm. (the hand being 1/7 of cubit or 74.88 mm.), so that its theoretical length should be 230.425 mm, and its actual length is 172 mm. less. Considering Cole's statement about the margin of possible error in his figures, it must be concluded that the sides were about 200 mm. shorter than their theoretical length. This difference must be ascribed to a mechanical error of the builders in the operation of measuring rather than to an imperfection in the ruler employed. Cole, as an experienced surveyor, calls attention to the difficulty in proceeding in a perfectly straight line in stretching a tape for a length of about 230 m. Another factor that would reduce the actual length is any imperfection in the process of making level the surface on which the measures were taken; but in this respect the builders were amazingly successful since there is an inclination of only 15 mm. from SE corner to the NW one.

4. Cole reports that the error in the construction of a right angle at the four corners of the Pyramid is the following:

NW corner	-0’ 2”
SW corner	+0’ 33”
SE corner	-3’ 33”
NE corner	+3’ 2”

The error in making these perpendiculars was 2” and 33”. Greater perfection was achieved in drawing the North side to which greater attention was paid. The North side not only presented special problems of measurement, but is also the most important side of the Pyramid, being directed towards the seat of the gods.

Borchardt noted that on the North side there is cut on the foundation blocks a line marking the middle point and assumed that this line indicated the main axis of the Pyramid. But Cole found that this line is 71 mm. closer to the NW corner than the middle point. Reginald Engelbach in his report about the Cole survey states: “This line was thus probably the original line of the axis.”

The figure 71 mm. probably corresponds to a hand or 74.88 mm.; the difference may be explained by a small error in determining the exact position of the NW corner.

The direction of the sides is the following:

West side	0º 2’ 30” W of true N
South side	0º 1’ 57” N of true E
East side	0º 5’ 30” W of true N
North side	0º 2’ 28” N of true E.

The figures indicate that the North and South sides were intended to be perpendicular to the West side whereas the East side was constructed as being at an angle of 3’ with the azimuth of the three other sides. The precision of the other figures intimates that the builders could not have erred by such an amount.

If the angle is 3’ the North side should be shorter by about 2 2/3 hands or more exactly 2.61 hands. Possibly this difference in the length of the North side was split, making the distance from the NW corner to the axis equal to 1540-1 hand and the distance from the axis to the NE corner to 1540-1 2/3 hands. But if Cole is correct in reporting that the shortening of the Western half side is 71 mm., a compromise was arrived at in marking the axis by setting it at an angle of 1’ with the West side, whereas the East side is at an angle of 3”. If the shortening is 74.88 mm. or a hand, the angle of the axis with the West side is 1’7”, assuming that the axis ends at the middle point of the South side.

5. The same contraction of the North side occurs in the Second Pyramid, the one constructed by Cheops’ successor. According to Petrie’s survey the sides have the following lengths:

West	215,277 mm.
South	215,313
East	215,269
North	215,186

It is agreed that the sides were computed as 410 cubits. The cubit employed here is that of 525 mm., so that the theoretical length is 215.250 mm. The builders succeeded in an almost perfect measurement of the sides: it may be that they had improved their techniques since the builders of the Great Pyramid, or more likely that they were aided by the possibility of measuring the diagonal lines. It was impossible to test the diagonals in the Great Pyramid, because it was built around a core of rock that was left in its natural state and not removed.

Concerning the orientation of the sides of the Second Pyramid, Petrie reports the following figures:

West	0°4'21” W of true N
South	0°5'40” N of true E
East	0°6'13” W of true N
North	0°5'31” N of true E

Petrie warns that the triangulation of Egypt existing at his time did not allow him to determine the North with absolute precision. By comparing his figures for the Great Pyramid with Cole’s, I would guess that his North was about 1’ West of true North. Hence, most tentatively I would suggest that Petrie’s figures may be corrected to read as follows:

West	0º3'21”
South	0º4'40”
East	0º5'13”
North	0º4'31”

In any case it is clear that the East side forms an angle of 1’52” with the West side. The North and the South sides are perfectly parallel. In the case of the Great Pyramid the North and South sides were drawn as perpendicular to the East side, whereas here the technique was improved by making the North and South sides intermediary between being perpendicular to the East side and being perpendicular to the West side. The exact intermediary value of the North and South sides would be 52” more than the inclination of the East side, so that, if Petrie’s values are absolutely exact, the difficult procedure of bisecting the small angle between the East side and the West side was carried through with an error of 28” and 37”. The procedure probably was that of marking the axis of the Pyramid as intermediary between that West side and the East side, as it was done in the Great Pyramid; then the North and the South sides were marked as perpendicular to the axis so marked. The perfection achieved in marking these perpendiculars was such that there is an angle of only 9” between the North and the South side.

Unfortunately an accurate survey of the dimensions of the Third Pyramid was never undertaken. The lack of precise reports about later pyramids may not be too regrettable, since they are all substantially smaller, so that the contraction of the North side, if it exists, would be hardly noticeable.

Three great pyramids were built by Cheops' predecessor Snefru, the founder of the Fourth Dynasty; but in these pyramids the true pyramidal form was experimented with for the first time, so that they may not reveal all the refinements of the Great Pyramid. Accurate data are available only for one of Snefru's pyramids, that of Meidun, but this pyramid was not oriented to the North and, furthermore, its pyramidal form was achieved only in a second moment by adding an outer cover to an original step pyramid.

6. The differences among the orientation of the sides are so small that in my opinion they may be explained only by the phenomenon of the precession of the equinoxes. The Pole moves to the West at a rate of 50".26 a year.

The question of the method followed in orienting the pyramids has been the object of a detailed study by Zbyněk Žába. The documents prove beyond any doubt that the initial operation in erecting an important structure in Egypt was the ceremony of the "stretching of the cord," by which through the observation of stars with some sort of transit there was determined the North-South direction. The East-West direction was marked by tracing a perpendicular to the basic line.

Žába tries to determine the stars used in the pointing by examining Egyptian charts of constellations. But exception must be taken to this approach because the orientation of the two pyramids of Gizah is too precise to be considered from this point of view. Charts of constellations would give only a good practical approximation to the true North. For instance, today we consider that α Ursae Minoris is the polar star and that the line passing by α and β Ursae Majoris gives the northerly direction; but these are approximations based on the reference points provided by the most visible stars. By such a method there would not be achieved a precision corresponding to that of the Pyramids. Charts of constellations are concerned only with the positions of the most important stars which can be readily identified without a pointing instrument.

Žába observes that since the pyramids were oriented to the North by the observation of stars, the position of the Pole must have been obtained by bisecting the angle formed by the two extreme positions of a circumpolar star. Most scholars are of the opinion that the orientation was obtained by this procedure or by a similar one. But a few scholars have observed that a much simpler method of obtaining the northerly direction is to observe the direction of the shortest shadow of the sun at the solstices. The bisection of the angle formed by two positions of a circumpolar star involves many possibilities of error. First of all the process of bisecting an angle exactly is difficult. Next, the process of determining which positions are opposite involves the use of clocks or an artificial line that is perfectly level. Some scholars, being aware of these possibilities of error, have suggested that there was observed the lowest culmination of a circumpolar star, which would give the North directly. But it is difficult to observe the exact point of the lowest culmination, because near this point the star moves almost horizontally; furthermore, the impact of refraction would be great in the observation of a circumpolar star at its lowest culmination, since the star would be at a narrow angle with the horizon.

The texts indicate that written instructions were followed in proceeding to “the stretching of the cord” and in drawing on the ground the outline of an important construction. I have determined that in Mesopotamia there was a division of roles between the mathematician who planned the structure and that of the builders a similar distinction may have been made in Egypt. I would suggest that, when the plan was prepared, as part of it there were performed calculations aimed at deciding which alignment of stars would give the North; the practical builders would not be concerned with astronomical problems, as they would not be concerned with the mathematical computations about the proportions of the structure, but they would simply point to specific stars according to the instructions.

If my hypothesis is correct, it would follow that in the case of the Great Pyramid, the West side is at angle of 2’30” with true North because exactly three years had passed between the formulation of the project and the moment in which the area of the West side was cleared and smoothed down for the marking of the base side. A period of three and half years would have passed before the East side was ready for the corresponding operation. In the case of the Second Pyramid, the interval would be three years between the stretching of the cord on the West side and the stretching of the cord at the East side. Unfortunately the datum about the absolute orientation of the West side, as reported by Petrie, is not entirely reliable; but if my guess that the angle of the West side was 3’21” is correct, four years would have passed between the formulation of the project and the drawing of the line of the West side.

7. These figures not only indicate the rate of speed at which the two Pyramids were constructed, but also illuminate important astronomical issues. First of all, computing the yearly precession as 50" or 50½", all figures fit, indicating that angles could be computed within a range of approximation of 2". Father F. X. Kugler, as an expert of Mesopotamian astronomy, has claimed that the unit of 2" that occurs in cuneiform astronomical texts could not have had any concrete application, because the ancients could not have even approached such a precision in angular measures, but he has never investigated problems of ancient metrology and techniques of measurement.

Since the several measurements of the sides seem to be at intervals of years or half years, it must be concluded that the orientation was based on the movement of the sun. It can be objected that the pointing to stars does not need to wait for the occurrence of equinoxes or solstices; but it may be supposed that the planners determined the proper alignment of stars by observing the sun at the solstices, and that the operation of stretching the cord was performed at the solstices, either because this was the traditional date for orientations, or in order to control exactly the angles by computing the effect of the precession.